

# Twisted Kodaira-Spencer classes and the geometry of surfaces of general type

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*To Fedor Bogomolov on his 65th birthday*

## ABSTRACT

We study the cohomology groups  $H^1(X, \Theta_X(-mK_X))$ , for  $m \geq 1$ , where  $X$  is a smooth minimal complex surface of general type,  $\Theta_X$  its holomorphic tangent bundle, and  $K_X$  its canonical divisor. One of the main results is a precise vanishing criterion for  $H^1(X, \Theta_X(-K_X))$  (Theorem 1.1).

The proof is based on the geometric interpretation of non-zero cohomology classes of  $H^1(X, \Theta_X(-K_X))$ . This interpretation in turn uses higher rank vector bundles on  $X$ .

We apply our methods to the long standing conjecture saying that the irregularity of surfaces in  $\mathbb{P}^4$  is at most 2. We show that if  $X$  has prescribed Chern numbers, no irrational pencil, and is embedded in  $\mathbb{P}^4$  with a sufficiently large degree, then the irregularity of  $X$  is at most 3.

*MSC.* 14J60, 14J29, 14F17

## 1 INTRODUCTION

From Kodaira-Spencer theory of deformations of complex structures, [10], it is known that for a smooth complex projective variety  $X$  with holomorphic tangent bundle  $\Theta_X$ , the cohomology group  $H^1(X, \Theta_X)$  parametrizes the infinitesimal deformations of complex structures on  $X$ —hence the importance to study this group. In this paper we propose to study its twisted versions. More precisely, let  $X$  be a smooth minimal complex surface of general type and consider the cohomology groups  $H^1(X, \Theta_X(-mK_X))$ , where  $m$  is a positive integer and  $K_X$  the canonical divisor of  $X$ . The basic question we wish to address is the vanishing (respectively the non-vanishing) of these groups. It is customary to call elements of  $H^1(X, \Theta_X)$  Kodaira-Spencer classes. Following this tradition, we will refer to classes of  $H^1(X, \Theta_X(-mK_X))$  as twisted Kodaira-Spencer classes of degree  $m$ .

Of course, if  $K_X$  is ample the general result of Serre on sheaf cohomology implies the vanishing of  $H^1(X, \Theta_X(-mK_X))$  for all  $m$  sufficiently big. On the other hand, if  $K_X$  is not ample, then  $X$  carries  $(-2)$  curves, *i.e.* smooth rational curves  $C$  with  $C^2 = -2$  and  $C \cdot K_X = 0$ . Furthermore, it is easy to see that every  $(-2)$  curve on  $X$  gives rise to a non-zero class in  $H^1(X, \Theta_X(-mK_X))$ , for *all* positive  $m$  (see (b) in Remark 2.5). This observation indicates that the twisted Kodaira-Spencer classes contain a rather precise geometric data. In fact our main line of investigation as well as the strategy for proving the

vanishing are very much modeled on this observation. Namely, we seek to extract geometric data from non-zero classes in  $H^1(X, \Theta_X(-mK_X))$ . Once this is done, we impose suitable hypotheses on  $X$  to rule out the existence of such data and thus obtain the vanishing of  $H^1(X, \Theta_X(-mK_X))$ . The following is one of the main results of the paper illustrating our approach.

**THEOREM 1.1.** *Let  $X$  be a smooth minimal complex surface of general type such that its Chern numbers satisfy  $c_2(X)/K_X^2 < 5/6$ . If  $X$  contains no smooth rational curves with self-intersection  $-2$  or  $-3$ , then*

$$H^1(X, \Theta_X(-K_X)) = 0.$$

This result points to the following general paradigm regarding the groups  $H^1(X, \Theta_X(-mK_X))$ , for  $m \geq 1$ :

*If the topological invariants of  $X$  are suitably constrained, then the non-vanishing of  $H^1(X, \Theta_X(-mK_X))$  implies the existence of configurations of special curves on  $X$ .*

This heuristic principle is realized in case  $m = 1$  by the following technical result which constitutes the backbone of our considerations and might be of independent interest.

**PROPOSITION 1.2.** *Let  $X$  be a smooth minimal complex surface of general type such that its Chern numbers satisfy  $c_2(X)/K_X^2 < 1$ . If  $H^1(X, \Theta_X(-K_X)) \neq 0$ , then the canonical divisor  $K_X$  admits a decomposition*

$$K_X = L + E \tag{1.1}$$

*subject to the following conditions:*

- (i)  *$L$  has the Iitaka dimension 2, and*
- (ii)  *$E$  is a non-zero effective divisor which is numerically effective (nef), unless  $E$  contains irreducible components  $C$  such that  $C \cdot E = -2$  or  $-1$ . Such components are smooth rational curves of self-intersection  $-2$  or  $-3$ .*

A way to prove Theorem 1.1 can now be summarized as follows: The hypotheses of Theorem 1.1 impose severe restrictions on the decomposition in (1.1) and these restrictions, in their turn, lead to a contradiction. Hence  $H^1(X, \Theta_X(-K_X))$  must vanish.

Now, we will explain how to pass from the non-vanishing of  $H^1(X, \Theta_X(-K_X))$  to the decomposition in (1.1) and to its various geometric properties. The main idea is to use the identification

$$H^1(X, \Theta_X(-K_X)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X)),$$

where  $\Omega_X$  is the cotangent bundle of  $X$  (see [14] for a similar approach). We can and will view a non-zero class  $\xi \in H^1(X, \Theta_X(-K_X))$  as the corresponding extension, *i.e.* the short exact sequence of locally free sheaves on  $X$

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \tag{1.2}$$

Our attention shifts to the middle term sheaf  $\mathcal{T}_\xi$ . It is locally free of rank 3, with Chern invariants

$$c_1(\mathcal{T}_\xi) = 0 \quad \text{and} \quad c_2(\mathcal{T}_\xi) = c_2 - K_X^2.$$

The hypothesis  $c_2(X)/K_X^2 < 1$  yields that  $\mathcal{T}_\xi$  is unstable in the sense of Bogomolov. Considering its *Bogomolov destabilizing subsheaf* is the key to obtaining the decomposition in (1.1). This subsheaf, which we call  $\mathcal{F}$ , must be of rank 2, and its determinant provides the divisor  $L$  in the decomposition (1.1). To see the other part of the decomposition, we observe that the inclusion of  $\mathcal{F}$  in  $\mathcal{T}_\xi$  combined with the epimorphism in (1.2) induces a morphism

$$\mathcal{F} \longrightarrow \Omega_X$$

which is generically an isomorphism. Furthermore its rank must drop along a non-zero divisor. This is the divisor  $E$  in (1.1). With the decomposition established, it is rather straightforward to obtain some information on the numerical properties of the decomposition, such as  $L^2$ ,  $L \cdot K_X$ ,  $E^2$ ,  $E \cdot K_X$  (see (2.2)). But one can go further on by getting rather detailed information concerning the irreducible components of  $E$ . This constitutes the most technical part of the paper.

The above discussion clearly shows that the non-zero twisted Kodaira-Spencer classes contain quite precise geometric information and one might be tempted to say that for understanding the geometry of  $X$ , it is more useful to have the non-vanishing of the groups  $H^1(X, \Theta_X(-mK_X))$  for some  $m > 0$ , rather than their vanishing.

Theorem 1.1 belongs to the category of “effective vanishing theorems”. It could also be viewed as a cohomological criterion for the existence of particular rational curves on surfaces of general type. This, in itself, can be quite helpful. At the same time, the groups of the more general form  $H^1(X, \Theta_X(-D))$ , with  $D$  a “positive” divisor, come up in various contexts. So the control of these groups can be useful for applications.

One immediate application comes from the deformation theoretic interpretation of  $H^1(X, \Theta_X(-D))$ . Similar to the usual Kodaira-Spencer classes, the twisted ones in  $H^1(X, \Theta_X(-D))$  can also be viewed as infinitesimal deformations. Namely, one considers deformations of pairs  $(X, C)$ , where  $X$  is a surface of general type and  $C$  is a smooth curve in the linear system  $|D|$ . The group  $H^1(X, \Theta_X(-D))$  parametrizes the infinitesimal deformations of  $(X, C)$  where the corresponding infinitesimal deformation of the curve  $C$  is trivial. In other words  $H^1(X, \Theta_X(-D))$  can be identified with the Zariski tangent space along the fibre through  $(X, C)$  of the forgetful functor

$$F : (X, C) \mapsto C.$$

So the vanishing of  $H^1(X, \Theta_X(-D))$  would tell us that this functor gives an immersion between the corresponding stacks. Thus Theorem 1.1 yields the immersion of  $\mathfrak{M}_{K^2, c_2}^p$ , the stack of pairs  $(X, C)$ , where  $C$  is a smooth curve in the canonical linear system  $|K_X|$  and  $X$  is subject to the hypotheses of Theorem 1.1, into  $\mathfrak{M}_{K^2+1}$ , the stack of smooth projective curves of genus  $g_C = K_X^2 + 1$ .

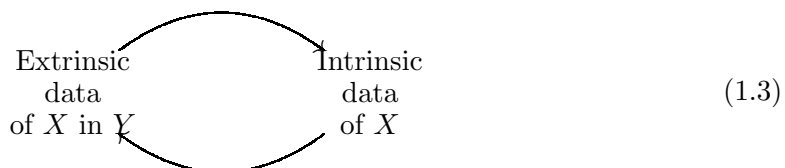
**COROLLARY 1.3.** *If  $\frac{c_2}{K^2} < \frac{5}{6}$ , then the forgetful functor  $F : \mathfrak{M}_{K^2, c_2}^p \rightarrow \mathfrak{M}_{K^2+1}$  is an immersion if the canonical system  $|K_X|$  contains a smooth curve.*

The second application developed in the paper comes from the general observation that the groups of the form  $H^1(X, \Theta_X(-D))$  ( $D$  is, as before, some positive divisor on  $X$ ) control a certain amount of the *extrinsic* geometry of  $X$ . More precisely, if  $X$  is embedded in a smooth projective variety  $Y$ , then the normal bundle  $\mathcal{N}_{X/Y}$  of  $X$  in  $Y$  is related to the tangent bundle  $\Theta_X$  of  $X$  via the normal sequence. Hence the relation between  $H^0(X, \mathcal{N}_{X/Y}(-D))$  and  $H^1(X, \Theta_X(-D))$ .

It may happen that for some *a priori* “extrinsic” reasons, one knows that the cohomology group  $H^0(X, \mathcal{N}_{X/Y}(-D)) \neq 0$ . Then, provided that the coboundary map

$$H^0(X, \mathcal{N}_{X/Y}(-D)) \longrightarrow H^1(X, \Theta_X(-D))$$

is not zero, one obtains the non-vanishing of  $H^1(X, \Theta_X(-D))$ . This, according to our heuristic principle, should impose topological and geometrical constraints on  $X$ . It potentially opens a way to having restrictions on the topology/geometry of surfaces embeddable in a given smooth projective variety  $Y$ . We summarize this “extrinsic – intrinsic” reasoning in the following diagram.



We apply this line of thinking to surfaces in  $\mathbb{P}^4$  and in particular to the long standing conjecture about the upper bound on the irregularity of surfaces in  $\mathbb{P}^4$ . This says that the irregularity of such surfaces is at most 2. For a more ample discussion of the history and the background of this problem we refer the reader to [13] and the references therein. To our understanding, there is no conceptual reason for such an upper bound and this estimate is largely based on the lack of examples. However, by exploiting non-zero twisted Kodaira-Spencer classes for appropriate twists of  $\Theta_X$ , we are able to show the following.

**THEOREM 1.4** (= Theorem 5.1). *Given a positive integer  $\chi$  and a rational  $\beta$ ,  $2 \leq \beta \leq 9$ , there is a number  $d_0(\chi, \beta)$  such every smooth minimal surface  $X \subset \mathbb{P}^4$  of degree  $d > d_0(\chi, \beta)$ , with  $\chi(\mathcal{O}_X) = \chi$ ,  $K_X^2 = \beta\chi(\mathcal{O}_X)$ , and having no irrational pencil, has the irregularity at most 3. Furthermore, if the irregularity is equal to 3, then  $X$  must be subject to the condition (i) or (ii) in Lemma 5.4.*

Let us explain how this result and its proof fit into the general heuristic scheme (1.3). To begin with, the constant  $d_0(\chi, \beta)$  and the condition  $d > d_0(\chi, \beta)$  of Theorem 1.4 provide an *a priori* extrinsic piece of information for the pair  $(X, \mathbb{P}^4)$  on the left side of the diagram (1.3). This datum is the existence of a 3-fold of degree  $m_X \leq 5$  in  $\mathbb{P}^4$  containing  $X$ . This observation is essentially due to Ellingsrud and Peskine in [6]. Then, the right oriented arrow in (1.3) translates the existence of this 3-fold into the intrinsic datum of the non-vanishing of  $H^1(X, \Theta_X(-D))$ , where  $D = K_X + (5 - m_X)H$ , and where  $H$  denotes the divisor class of hyperplane sections of  $X \subset \mathbb{P}^4$ . A version of Proposition 1.2 implies a decomposition, as in (1.1), of the canonical divisor. It should be pointed out that in this version we use the hypothesis on irrational pencils instead of the inequality between the Chern numbers of Proposition 1.2. The intrinsic datum of the decomposition of  $K_X$  is then translated back into extrinsic data by establishing a relationship between  $L$  in (1.1)

and the hyperplane section  $H$ —this is the left oriented arrow in (1.3). Our bound on the irregularity of  $X$  is an immediate consequence of this relation.

Though Theorem 1.4 is a long way from the unconditional bound of the conjecture, it seems to be useful since it suggests that one of the conceptual reasons for not having surfaces in  $\mathbb{P}^4$  with large irregularity is the *non-vanishing* of cohomology groups of the form  $H^1(X, \Theta_X(-D))$  for an appropriately chosen “positive” divisor  $D$  on  $X$ .

The arguments developed in the proof of Theorem 1.4 can also be used to show that for high degree surfaces in  $\mathbb{P}^4$  with bounded holomorphic Euler characteristic, their topological index is negative. (See Theorem 5.5 for the precise statement.)

Let us end this section with some comments and questions. Theorem 1.1 together with the additional assumption that the canonical linear system  $|K_X|$  contains a reduced irreducible member implies the vanishing of  $H^1(X, \Theta_X(-mK_X))$  for all  $m \geq 1$  (Corollary 4.4). This gives a complete answer to the question about the vanishing of the groups  $H^1(X, \Theta_X(-mK_X))$ ,  $m \geq 1$ , for a large class of surfaces of general type. However, the situation is not fully satisfactory with regard to the hypothesis in Theorem 1.1 concerning the  $(-3)$ -curves—we do not know if this assumption is really necessary. Let us explain this point. Contrary to  $(-2)$ -curves, the  $(-3)$ -curves, by themselves, do not produce cohomology classes in  $H^1(X, \Theta_X(-K_X))$ . These curves appear naturally in the course of the study of the decomposition of  $K_X = L + E$  in (1.1). But each  $(-3)$ -curve must appear in the divisor  $E$  as a part of a rather involved configuration of other curves, and at this stage, we are unable to treat these configurations efficiently.

It should also be noticed that the problem of  $(-3)$ -curves disappears for the groups  $H^1(X, \Theta_X(-mK_X))$ , with  $m \geq 4$ , and our approach allows to have a purely cohomological criterion for the existence of  $(-2)$ -curves on surfaces of general type (that will appear elsewhere).

We also believe that the groups  $H^1(X, \Theta_X(-mK_X))$ , for all  $m$  sufficiently large, must be spanned by the twisted Kodaira-Spencer classes associated to  $(-2)$ -curves. Another way to put it, the twisted Kodaira-Spencer classes of sufficiently high degree should provide a stronger version of the decomposition of type (1.1) where the divisor  $E$  is composed entirely of  $(-2)$ -curves.

We limited our discussion to surfaces of general type. However, it is clear that our approach is valid in higher dimensions as well and one can ask if it is possible to have a result analogous to Theorem 1.1 in dimensions bigger than 2.

The paper is organized as follows. In §2, we prove the decomposition (1.1) and derive the basic properties of the divisor  $L$  in this decomposition (see Lemma 2.1 and Lemma 2.2). The third section is devoted to a detailed study of the irreducible components of the divisor  $E$  in (1.1). In §4, we give a proof of Theorem 1.1. Finally, in §5, the two results about surfaces in  $\mathbb{P}^4$  are considered (see Theorem 5.1 and Theorem 5.5).

#### NOTATION AND CONVENTIONS.

$X$  is a smooth minimal complex surface of general type unless otherwise stated.

$K_X$  is the canonical divisor of  $X$ .

$K_X^2$ ,  $c_2(X)$ , and  $\chi(\mathcal{O}_X)$  are the Chern numbers and the holomorphic Euler characteristic of  $X$  respectively.

$$\alpha_X := \frac{c_2(X)}{K_X^2}.$$

$\Theta_X$  is the holomorphic tangent bundle of  $X$  and  $\Omega_X$  is its holomorphic cotangent bundle.  $p_g(X) = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X))$  and  $q(X) = h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X)$  are respectively, the geometric genus and the irregularity of  $X$ .

For a coherent sheaf  $\mathcal{G}$  on a variety  $M$ , the cohomology group  $H^i(M, \mathcal{G})$  is denoted by  $H^i(\mathcal{G})$  if no confusion is likely.

For an effective divisor  $D = \sum_{\alpha} \mu_{\alpha} C_{\alpha}$  on a surface  $X$ , where the  $C_{\alpha}$  are reduced irreducible curves,  $D^{\text{red}}$  denotes the reduced associated divisor  $\sum_{\alpha} C_{\alpha}$ .

All equalities between divisors take place in  $A^1(X)$  of the Chow ring of  $X$  except the ones involving the Zariski decomposition. In these cases, the equalities are considered in the Néron-Severi group  $NS(X)_{\mathbb{Q}} = NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

## 2 THE NON-VANISHING OF $H^1(X, \Theta_X(-K_X))$ AND THE DECOMPOSITION OF $K_X$

In this section we show how the existence of a non-trivial element in  $H^1(X, \Theta_X(-K_X))$  gives rise to a distinguished decomposition of the canonical divisor  $K_X$  of  $X$ .

LEMMA 2.1. *Assume the Chern numbers  $K^2$  and  $c_2(X)$  are subject to*

$$\alpha_X = \frac{c_2(X)}{K_X^2} < 1$$

*and assume  $H^1(X, \Theta_X(-K_X))$  to be non-zero. Then the canonical divisor  $K_X$  admits the decomposition*

$$K_X = L + E$$

*where  $L$  has Iitaka dimension 2 and  $E$  is effective and non-zero.*

*Proof.* A non-zero element  $\xi \in H^1(X, \Theta_X(-K_X))$  via the natural identification

$$H^1(X, \Theta_X(-K_X)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X))$$

defines the extension

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{T}_{\xi} \longrightarrow \Omega_X \longrightarrow 0.$$

The sheaf  $\mathcal{T}_{\xi}$ , seating in the middle of the sequence, is locally free of rank 3 and its Chern invariants are

$$c_1(\mathcal{T}_{\xi}) = 0 \quad \text{and} \quad c_2(\mathcal{T}_{\xi}) = c_2 - K_X^2.$$

The assumption  $\alpha_X < 1$  implies that  $\mathcal{T}_{\xi}$  is Bogomolov unstable. Let  $\mathcal{F}$  be its Bogomolov destabilizing subsheaf. One knows that its determinant  $\det \mathcal{F} = \mathcal{O}_X(L)$  is in the positive cone of  $NS(X)_{\mathbb{Q}}$  (see [3]). This and the fact that the cotangent sheaf  $\Omega_X$  cannot have subsheaves of rank 1 and of Iitaka dimension 2 imply that the rank of  $\mathcal{F}$  is 2. Furthermore, we may assume it to be saturated. Then the quotient  $\mathcal{T}_{\xi}/\mathcal{F}$  is torsion-free and  $\mathcal{T}_{\xi}/\mathcal{F} \simeq$

$\mathcal{J}_Z(-L)$ , where  $Z$  is a subscheme of codimension 2 and  $\mathcal{J}_Z$  is its sheaf of ideals. This yields the following diagram.

$$\begin{array}{ccccccc}
 & & & 0 & & & (2.1) \\
 & & & \downarrow & & & \\
 & & & \mathcal{F} & & & \\
 & & & \downarrow & \searrow \varphi_\xi & & \\
 0 & \longrightarrow & \mathcal{O}_X(-K_X) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X \longrightarrow 0 \\
 & & \searrow \psi_\xi & & \downarrow & & \\
 & & & & \mathcal{J}_Z(-L) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

Using again the fact that  $\Omega_X$  has no rank 1 subsheaves of Iitaka dimension 2, we conclude that the morphism  $\varphi_\xi$  in (2.1) is generically an isomorphism. In addition, its cokernel,  $\text{coker } \varphi_\xi$ , is a sheaf supported on a codimension 1 subscheme since otherwise  $\varphi_\xi$  would be an isomorphism making the extension class  $\xi$  trivial. The resulting exact sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi_\xi} \Omega_X \longrightarrow \text{coker } \varphi_\xi \longrightarrow 0$$

yields the asserted decomposition

$$K_X = L + E$$

with  $L$  of Iitaka dimension 2 and  $E = c_1(\text{coker } \varphi_\xi)$  effective and non-zero.  $\square$

We begin the study of the decomposition of  $K_X$  in Lemma 2.1 by recording some properties of  $L$ .

LEMMA 2.2. *The divisor  $L$  in Lemma 2.1 admits the Zariski decomposition<sup>1</sup>*

$$L = L^+ + L^-$$

where  $L^+$ , the positive part of  $L$ , is nef and big, and  $L^-$  is the negative part of  $L$ . Furthermore,  $L$  is subject to the following numerical conditions:

$$L^2 \geq \frac{3}{2}(1 - \alpha_X) K_X^2 \quad \text{and} \quad L \cdot K_X \geq \sqrt{\frac{3}{2}(1 - \alpha_X) K_X^2}. \quad (2.2)$$

*Proof.* By construction,  $L$  has Iitaka dimension 2, hence it has the Zariski decomposition  $L = L^+ + L^-$  as asserted.

The two inequalities of the lemma come from the diagram (2.1). Namely, combining its vertical and horizontal sequences yields

$$c_2(X) - K_X^2 = c_2(\mathcal{F}) - L^2 + \deg Z.$$

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<sup>1</sup>As we have specified in the introduction, the equality  $L = L^+ + L^-$  takes place in  $NS(X)_\mathbb{Q}$ .

Since  $\mathcal{F}$  is a rank 2 subsheaf of the cotangent sheaf  $\Omega_X$  and  $\det(\mathcal{F}) = \mathcal{O}_X(L)$  has Zariski decomposition, a result of Miyaoka, [12, Remark 4.18], says that its Chern invariants are subject to

$$c_2(\mathcal{F}) \geq \frac{1}{3}L^2 - \frac{1}{12}(L^-)^2.$$

Combining the two inequalities, we obtain

$$L^2 \geq \frac{3}{2}(K_X^2 - c_2(X)) + \frac{3}{2}\deg Z - \frac{1}{8}(L^-)^2 \geq \frac{3}{2}(1 - \alpha_X)K_X^2, \quad (2.3)$$

thus the first of the asserted inequalities. The second follows from the first and the Hodge index theorem  $(L \cdot K_X)^2 \geq K_X^2 L^2$ .  $\square$

The following lemma compares the two divisors,  $L$  and  $E$ , of the decomposition in Lemma 2.1.

LEMMA 2.3. *Let  $K_X = L + E$  be the decomposition in Lemma 2.1.*

- 1) *If  $\alpha_X < 5/6$ , then  $L \cdot K_X > E \cdot K_X$ .*
- 2) *If  $\alpha_X \leq 1/2$ , then  $(L - E)^2 \geq 0$ .*

*Proof.* To prove 1), assume  $L \cdot K_X \leq E \cdot K_X$ . Then,

$$L \cdot K_X \leq E \cdot K_X \leq K_X^2 - L \cdot K_X,$$

and using (2.2), we obtain

$$K_X^2 \geq 2L \cdot K_X \geq 2\sqrt{\frac{3}{2}(1 - \alpha_X)K_X^2}.$$

Hence  $1 \geq \sqrt{6(1 - \alpha_X)}$ , i.e.  $\alpha_X \geq 5/6$ .

To prove 2) we use the hypothesis  $\alpha_X \leq 1/2$  and (2.2) to arrive at

$$\begin{aligned} (L - E)^2 &= (2L - K_X)^2 = 4L^2 - 4L \cdot K_X + K_X^2 \geq 6(1 - \alpha_X)K_X^2 - 4L \cdot K_X + K_X^2 \\ &\geq 3K_X^2 - 4L \cdot K_X + K_X^2 = 4(K_X^2 - L \cdot K_X) = 4E \cdot K_X \geq 0. \end{aligned}$$

$\square$

In the sequel, we will need the following cohomological property of the divisor  $E$ .

LEMMA 2.4. *Let  $e$  be a section of  $\mathcal{O}_X(E)$  corresponding to the divisor  $E$ . Then  $e \cdot \xi = 0$  in  $H^1(\Theta_X(E - K_X))$ .*

*Proof.* Dualizing the diagram (2.1) and tensoring it with  $\mathcal{O}_X(-L)$  give the following diagram.

$$\begin{array}{ccccccc} & & \mathcal{O}_X & & & & (2.4) \\ & & \downarrow & \searrow e & & & \\ 0 & \longrightarrow & \Theta_X(-L) & \longrightarrow & \mathcal{T}_\xi^*(-L) & \longrightarrow & \mathcal{O}_X(K_X - L) \longrightarrow 0 \end{array}$$



We see that the coboundary map

$$H^0(\mathcal{O}_X(E)) \longrightarrow H^1(\Theta_X(-L)) = H^1(\Theta_X(E - K_X))$$

given by the cup-product with the class  $\xi$  contains the section  $e$  in its kernel.  $\square$

REMARK 2.5. (a) The equality  $e \cdot \xi = 0$  in Lemma 2.4 implies that  $\xi$  is supported on the divisor  $E = \{e = 0\}$ . More precisely, from the exact sequence

$$0 \longrightarrow H^0(\Theta_X \otimes \mathcal{O}_E(E - K_X)) \longrightarrow H^1(\Theta_X(-K_X)) \xrightarrow{e} H^1(\Theta_X(E - K_X))$$

it follows that  $\xi$  is the image of a unique global section of  $\Theta_X \otimes \mathcal{O}_E(E - K_X)$ .

(b) In case  $E$  is a  $(-2)$ -curve, one has a converse, *i.e.*  $E$  defines a unique (up to a non-zero scalar multiple) non-zero cohomology class in  $H^1(\Theta_X(-K_X))$ . This follows since

$$\Theta_X \otimes \mathcal{O}_E(E - K_X) = \Theta_X \otimes \mathcal{O}_E(E) \cong \Theta_X \otimes \mathcal{O}_{\mathbb{P}^1}(-2) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4),$$

and hence  $H^0(\Theta_X \otimes \mathcal{O}_E(E - K_X)) = H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-4)) = \mathbb{C}$ . The same argument is valid for all  $H^1(\Theta_X(-mK_X))$ , where  $m \geq 1$ . Thus, a  $(-2)$ -curve gives rise to a non-zero cohomology class in every group  $H^1(\Theta_X(-mK_X))$ ,  $m \geq 1$ .

### 3 THE STUDY OF THE IRREDUCIBLE COMPONENTS OF $E$

This section is entirely devoted to a study of the divisor  $E$  of the decomposition in Lemma 2.1.

LEMMA 3.1. *The divisor  $E$  passes through the subscheme  $Z$ .*

*Proof.* To see this, it suffices to show that  $h^0(X, \mathcal{J}_Z(E)) \neq 0$ . This follows by tensoring the morphism  $\psi_\xi$  in (2.1) with  $\mathcal{O}_X(K_X)$ .  $\square$

From the vertical sequence in (2.1), the subscheme  $Z$  can be seen as the zero-locus of a section  $s_\xi \in H^0(X, \mathcal{T}_\xi^*(-L))$ . Using this section, we can define the restriction of  $Z$  to a component of  $E$ . More precisely, for every component  $C$  of  $E$ , denote by  $Z_C = \{s_\xi|_C = 0\}$ .

The main tool for a detailed study of  $E$  is the following construction. We take the second exterior product in diagram (2.1) to obtain the diagram,

$$\begin{array}{ccccccc} & & \mathcal{O}_X(L) & & & & (3.1) \\ & & \downarrow & \searrow e & & & \\ 0 & \longrightarrow & \Theta_X & \longrightarrow & \bigwedge^2 \mathcal{T}_\xi & \longrightarrow & \mathcal{O}_X(K_X) \longrightarrow 0 \end{array}$$

where  $e$  is a section of  $\mathcal{O}_X(E)$  corresponding to the divisor  $E$ . For any component  $C$  of  $E$ , the morphism in (3.1) given by  $e$  vanishes over  $C$  and hence it induces a non-zero morphism

$$\tau_C : \mathcal{O}_C(L) \rightarrow \Theta_X \otimes \mathcal{O}_C$$

whose zero locus is  $Z_C$ . If  $C$  is a reduced and irreducible component of  $E$ , we combine this morphism with the normal sequence of  $C$  in  $X$  to obtain the following commutative diagram.

$$\begin{array}{ccccccc} & & & \mathcal{O}_C(L) & & & \\ & & & \downarrow \tau_C & \searrow \psi_C & & \\ 0 & \longrightarrow & \Theta_C & \longrightarrow & \Theta_X \otimes \mathcal{O}_C & \longrightarrow & \mathcal{O}_C(C) \end{array}$$

The irreducible components  $C$  of  $E$  will be distinguished according to the properties of the morphism  $\psi_C$ .

**type I:** If  $\psi_C = 0$ , in this case  $\tau_C$  factors through the tangent sheaf  $\Theta_C$  of  $C$  and the degree of  $Z_C$  is subject to

$$\deg Z_C \leq \deg \eta_C^*(Z_C) = 2 - 2g(\tilde{C}) - C \cdot L, \quad (3.2)$$

where  $\eta_C : \tilde{C} \rightarrow C$  is the normalization of  $C$  (see [14], pp.432-433, for details). Such a component will be called of type I.

**type II:** If  $\psi_C \neq 0$ , in this case  $\psi_C$  defines a non-zero section of  $\mathcal{O}_C(C - L)$  vanishing on  $Z_C$ , and hence

$$\deg Z_C \leq C^2 - C \cdot L. \quad (3.3)$$

We will call such a component of type II.

The divisor  $E$  can be written as the sum of two parts,

$$E = E_I + E_{II}, \quad (3.4)$$

with

$$E_I := \sum_{C \text{ of type I}} m_C C \quad \text{and} \quad E_{II} := \sum_{C \text{ of type II}} m_C C,$$

where  $m_C$  denotes the multiplicity of a component  $C$  in  $E$ .

The following lemma relates the components of  $E_{II}$  and the Zariski decomposition of  $L$  in Lemma 2.2 under the additional assumption  $\alpha_X < \frac{5}{6}$  of Theorem 1.1.

**LEMMA 3.2.** *Let  $\alpha_X < \frac{5}{6}$ . Then every irreducible component  $C$  of  $E$  of type II must also be a component of  $L^-$  and*

$$C \cdot L \leq C^2 < 0.$$

*Furthermore, the multiplicity of  $C$  in  $L^-$  is  $\geq 1$ .*

*Proof.* From (3.3), we deduce  $C \cdot L \leq C^2$ . First, we claim that  $C \cdot L \leq 0$ . Indeed, if  $C \cdot L$  is positive, then  $C^2 > 0$ . From this and the Hodge index theorem,  $(C \cdot L)^2 \geq C^2 L^2$ , it follows that

$$C^2 \geq C \cdot L \geq L^2 \geq \frac{3}{2} (1 - \alpha_X) K_X^2 \quad (3.5)$$

where the last inequality comes from Lemma 2.2. This and the Hodge index  $(C \cdot K_X)^2 \geq C^2 K_X^2$  yield

$$E \cdot K_X \geq C \cdot K_X \geq \sqrt{\frac{3}{2}(1 - \alpha_X)} K_X^2.$$

Combining this with the second inequality of Lemma 2.2 we obtain

$$K_X^2 = L \cdot K_X + E \cdot K_X \geq 2 \sqrt{\frac{3}{2}(1 - \alpha_X)} K_X^2.$$

But this yields  $\alpha_X \geq 5/6$ , contrary to our assumption.

Next combining the inequality  $C \cdot L \leq 0$  with the Zariski decomposition  $L = L^+ + L^-$  we obtain that  $C^2 < 0$ . This, together with  $L \cdot C \leq C^2$ , yields  $L \cdot C < 0$ . Hence  $C$  must be a component of  $L^-$ .

Turning to the last assertion of the lemma, let  $\mu_C$  be the multiplicity of  $C$  in  $L^-$ . Then

$$0 \geq C \cdot L - C^2 = C \cdot (L^+ + L^-) - C^2 = C \cdot L^- - C^2 \geq (\mu_C - 1)C^2,$$

hence  $\mu_C \geq 1$ . □

In analyzing the components of type I, we will repeatedly make use of the following lemma.

**LEMMA 3.3.** *The singular locus of  $E_I^{\text{red}}$  is contained in  $Z_{E_I^{\text{red}}}$ . (Recall, that for an effective divisor  $D = \sum_{\alpha} \mu_{\alpha} C_{\alpha}$ , where  $C_{\alpha}$ 's are reduced irreducible curves,  $D^{\text{red}}$  denotes the reduced associated divisor  $\sum_{\alpha} C_{\alpha}$ .)*

*Proof.* Taking the restriction of the diagram (3.1) to  $E_I^{\text{red}}$  implies that, outside  $Z_{E_I^{\text{red}}}$ , the sheaf  $\Theta_{E_I^{\text{red}}}$  is isomorphic to the locally free sheaf  $\mathcal{O}_{E_I^{\text{red}}}(L)$ . By a result of Lipman, [11], this implies that  $E_I^{\text{red}}$  is smooth outside the subscheme  $Z_{E_I^{\text{red}}}$ . □

Next we turn to the Zariski decomposition of  $E$ . We first determine the curves intersecting  $E$  negatively.

**LEMMA 3.4.** *Assume  $K_X$  to be ample. If  $C$  is a reduced irreducible curve on  $X$  such that  $C \cdot E < 0$ , then  $C$  is a smooth rational component of  $E_I$  satisfying  $C^2 = -3$ ,  $C \cdot E = -1$ , and  $C \cdot L = 2$ . Furthermore,  $C$  does not intersect any other component of  $E_I$ .*

*Proof.* The inequality  $C \cdot E < 0$  implies that  $C$  is a component of  $E$  and that  $C^2 < 0$ . Since  $K_X$  is ample,

$$C \cdot L = C \cdot (K_X - E) \geq 2, \tag{3.6}$$

so by (3.3),  $C$  must be a component of type I and according to (3.2),  $C \cdot L \leq 2$ . This inequality and (3.6) yield  $C \cdot L = 2$  and  $Z_C = \emptyset$ . The latter, combined with Lemma 3.3, implies that  $C$  is a smooth rational curve with  $C \cdot K_X = 1$ , i.e.  $C$  is a  $(-3)$ -curve, and that  $C$  intersects no other component of  $E_I$ . □

**REMARK 3.5.** If we drop the assumption of ampleness for  $K_X$ , then the inequality in (3.6) implies that the intersection  $C \cdot E$  is either  $-2$  or  $-1$ . The first case implies that  $C$  is a  $(-2)$ -curve, while the second leads to two possibilities: either  $C \cdot L = 1$  and  $C$  is again a  $(-2)$ -curve, or  $C \cdot L = 2$  and  $C$  is a  $(-3)$ -curve.

PROPOSITION 3.6. *Let  $K_X$  be ample and set  $\mathfrak{R}_{-1} = \{C \mid C \cdot E = -1\}$ . Then the Zariski decomposition of  $E$  is  $E = E^+ + E^-$ , where*

$$E^- = \frac{1}{3} \sum_{C \in \mathfrak{R}_{-1}} C.$$

*In particular,  $E^+ \neq 0$ .*

*Proof.* We know that  $E \neq 0$ . Let  $E^+ = E - E^-$ , with  $E^-$  defined as above. It is sufficient to show that  $C \cdot E^+ = 0$ , for any  $C \in \mathfrak{R}_{-1}$ , and that  $E^+$  is nef. The non-triviality of  $E^+$  then follows immediately.

We begin by checking that the curves in  $\mathfrak{R}_{-1}$  are orthogonal to  $E^+$ . To do this, we observe that if  $\Gamma \in \mathfrak{R}_{-1}$ , then  $\Gamma \cdot L = 2$ ; from (3.2), it follows that the subscheme  $Z_\Gamma$  is empty. Using Lemma 3.3, we see that  $\Gamma$  intersects no other curve from  $\mathfrak{R}_{-1}$ , hence

$$\Gamma \cdot E^+ = \Gamma \cdot \left( E - \frac{1}{3} \sum_{C \in \mathfrak{R}_{-1}} C \right) = \Gamma \cdot E - \frac{1}{3} \Gamma^2 = 0.$$

Next we show that  $E^+$  is nef. Let us suppose that there exists a reduced irreducible curve  $\Gamma$  such that  $\Gamma \cdot E^+ < 0$ . By the above argument,  $\Gamma \notin \mathfrak{R}_{-1}$  and

$$0 \leq \Gamma \cdot E = \Gamma \cdot E^+ + \Gamma \cdot E^- < \Gamma \cdot E^-.$$

Putting together this inequality and the last assertion of Lemma 3.4, we deduce that  $\Gamma$  is a type II component of  $E$ . The rest of the argument is an estimate of  $\Gamma \cdot E$ . To begin with,

$$\Gamma \cdot E = \Gamma \cdot (E_I + E_{II}) \geq \Gamma \cdot E_I + m_\Gamma \Gamma^2,$$

where  $m_\Gamma$  is the multiplicity of  $\Gamma$  in  $E$ . Using (3.3),

$$\Gamma \cdot E \geq \Gamma \cdot E_I + m_\Gamma (\Gamma \cdot L + \deg Z_\Gamma) \geq \Gamma \cdot E_I + m_\Gamma \Gamma \cdot K_X - m_\Gamma \Gamma \cdot E,$$

hence

$$\Gamma \cdot E \geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X + \frac{1}{m_\Gamma + 1} \Gamma \cdot E_I \geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X + \frac{1}{m_\Gamma + 1} \sum_{C \in \mathfrak{R}_{-1}} m_C \Gamma \cdot C.$$

Since by Lemma 3.4, every  $C \in \mathfrak{R}_{-1}$  intersects no other curve in  $E_I$ ,

$$-1 = C \cdot E = -3m_C + C \cdot E_{II} \geq -3m_C + m_\Gamma C \cdot \Gamma.$$

Thus

$$m_C \geq \frac{1}{3} \left( m_\Gamma \Gamma \cdot C + 1 \right).$$

Using this inequality and the last estimate for  $\Gamma \cdot E$ , we arrive at

$$\begin{aligned} \Gamma \cdot E &\geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X + \frac{1}{3} \frac{1}{m_\Gamma + 1} \sum_{C \in \mathfrak{R}_{-1}} (m_\Gamma \Gamma \cdot C + 1) \Gamma \cdot C \\ &\geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X + \frac{1}{3} \sum_{C \in \mathfrak{R}_{-1}} \Gamma \cdot C \\ &= \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X + \Gamma \cdot E^-, \end{aligned}$$

which leads to the contradiction

$$0 > \Gamma \cdot E^+ \geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X > 0.$$

□

REMARK. The preceding argument shows that every type II component  $\Gamma$  of  $E$  intersects  $E^+$  positively. More precisely,

$$\Gamma \cdot E^+ \geq \frac{m_\Gamma}{m_\Gamma + 1} \Gamma \cdot K_X$$

where  $m_\Gamma$  is the multiplicity of  $\Gamma$  in  $E$ .

Putting together Lemma 3.2 with the property (3.2) of curves of type I, we obtain the following.

PROPOSITION 3.7. *Assume  $\alpha_X < \frac{5}{6}$  and let  $C$  be an irreducible component of  $E$ . Then*

- *either  $C \cdot L \leq 0$  and then  $C$  is orthogonal to  $L^+$ , the positive part of the Zariski decomposition of  $L$ ,*
- *or  $C \cdot L > 0$  and then  $C$  is a smooth rational curve of type I with  $C \cdot L = 1$  or  $2$ .*

*Proof.* We consider  $C$  according to the sign of its intersection product with  $L$ . If  $C \cdot L \leq 0$ , we use the Zariski decomposition of  $L = L^+ + L^-$  in Lemma 2.2 to deduce that either  $C$  is a component of  $L^-$ , in which case  $C \cdot L^+ = 0$ , or it is not, in which case

$$0 \leq C \cdot L^+ = C \cdot (L - L^-) \leq -C \cdot L^- \leq 0,$$

hence again  $C \cdot L^+ = 0$ .

If  $C \cdot L > 0$ , then, by Lemma 3.2, the curve  $C$  is of type I, and by (3.2),

$$0 \leq 2g(\tilde{C}) \leq 2 - C \cdot L - \deg Z_C < 2.$$

This implies that  $C \cdot L \leq 2$  and  $g(\tilde{C}) = 0$ . Furthermore, by Lemma 3.3,  $C$  is smooth outside the subscheme  $Z_C$ , while the inequality (3.2),

$$\deg(Z_C) \leq \deg \eta_C^*(Z_C) \leq 1,$$

implies that  $Z_C$  is either empty or a smooth point of  $C$ . □

COROLLARY 3.8. *Assume  $K_X$  ample and  $\alpha_X < \frac{5}{6}$ . Then for every irreducible component  $C$  of  $E$ ,*

$$C \cdot L^+ \leq 1$$

*Furthermore, if  $C \cdot L^+ > 0$ , then  $C$  is a smooth rational curve of type I such that*

$$C \cdot L = j \quad \text{and} \quad C \cdot E_{II}^{\text{red}} = j - 1, \quad j \in \{1, 2\}.$$

*Proof.* Let  $C$  be an irreducible component of  $E$ . Using the previous proposition, we know that  $C \cdot L^+ = 0$  unless  $C$  is a smooth rational curve of type I such that  $C \cdot L = j$ ,  $j \in \{1, 2\}$ . Let  $C$  be such a rational curve with  $C \cdot L^+ > 0$ . Clearly  $C$  is not a component of  $L^-$ , hence  $C \cdot L^- \geq 0$ , and we have

$$C \cdot L^+ = j - C \cdot L^-. \quad (3.7)$$

In view of Lemma 3.2, if  $j = 1$ , the curve  $C$  cannot intersect any type II component of  $E$ , *i.e.*  $C \cdot E_{II}^{\text{red}} = 0$ .

We now turn to the case  $j = 2$ . First we will show that  $C \cdot E_{II} > 0$ . To see this, notice that

$$C \cdot E_{II} = C \cdot E - C \cdot E_I \geq -1 - C \cdot E_I, \quad (3.8)$$

where the last inequality uses Lemma 3.4. From  $C \cdot L = 2$  and the inequality (3.2), it follows that  $Z_C = \emptyset$ . Hence, according to Lemma 3.3,  $C$  intersects no other component of  $E_I$ , *i.e.*  $C \cdot E_I = m_C C^2$ , where  $m_C$  is the multiplicity of  $C$  in  $E_I$ . Substituting it in (3.8) leads to

$$C \cdot E_{II} = -1 - m_C C^2 > 0.$$

Thus, there are irreducible components  $\Gamma$  in  $E_{II}$  intersecting  $C$  non-trivially. Since, by Lemma 3.2, each such  $\Gamma$  appears in  $L^-$  with multiplicity  $\mu_\Gamma \geq 1$ , the equality (3.7) implies that there exists a unique  $\Gamma$  in  $E_{II}$  intersecting  $C$  and  $\Gamma \cdot C = 1$ . Hence  $C \cdot E_{II}^{\text{red}} = 1$  and  $C \cdot L^+ \leq 1$ .  $\square$

REMARK 3.9. From Corollary 3.8 it follows that the main reason for the non-vanishing of  $H^1(\Theta_X(-K_X))$  resides in the existence of curves  $C$  of type I with  $C \cdot L^+ > 0$ . Indeed, suppose that there would be no such curve. Then  $C \cdot L^+ = 0$  for every component  $C$  of  $E$ . This gives  $E^+ \cdot L^+ = 0$ . Since  $E^+$  is nef and non-zero, this is impossible.

#### 4 THE PROOF OF THEOREM 1.1

We now assume that  $K_X$  is ample and  $\alpha_X < 5/6$ . We seek to obtain a contradiction to the decomposition

$$K_X = L + E$$

of Lemma 2.1. Our main idea is to compare  $E^2$  and  $E \cdot L^+$ . This will be realized by using the properties of the divisors  $L$  and  $E$  accumulated in the previous sections. In view of Remark 3.9, we need to deal with the smooth rational curves  $C$  of type I subject to  $C \cdot L^+ > 0$ . By Corollary 3.8, these are components of  $E_I$  satisfying either  $C \cdot L = 1$ , or  $C \cdot L = 2$ . For this reason, we introduce the sets

$$\mathfrak{R}^j = \{C \mid C \subset E^+, C \cdot L = j\}, \quad j = 1, 2. \quad (4.1)$$

We have seen in Proposition 3.7 that every curve in  $\mathfrak{R}^j$  is smooth and rational. For any integer  $k$ , set

$$\mathfrak{R}_k^j = \{C \in \mathfrak{R}^j \mid C \cdot E = k\}.$$

Since every  $C \in \mathfrak{R}^j$  verifies  $0 < C \cdot K_X = j + C \cdot E$ , the sets  $\mathfrak{R}^j$  are partitioned by the subsets  $\mathfrak{R}_k^j$  as follows:

$$\mathfrak{R}^1 = \mathfrak{R}_0^1 \cup \mathfrak{R}_{\geq 1}^1 \quad \text{and} \quad \mathfrak{R}^2 = \mathfrak{R}_{-1}^2 \cup \mathfrak{R}_0^2 \cup \mathfrak{R}_{\geq 1}^2$$

where  $\mathfrak{R}_{\geq 1}^j = \bigcup_{k \geq 1} \mathfrak{R}_k^j$ .

REMARK.  $\mathfrak{R}_{-1}^2 = \mathfrak{R}_{-1}$ , cf. Lemma 3.4.

Using Proposition 3.7 and the above partitions of the sets  $\mathfrak{R}_j$ ,

$$E = Q + \sum_{C \in \mathfrak{R}_0^1} m_C C + \sum_{C \in \mathfrak{R}_0^2} m_C C + \sum_{C \in \mathfrak{R}_{-1}^2} m_C C, \quad (4.2)$$

where  $Q$  contains all the components lying in  $\mathfrak{R}_{\geq 1}^j$  ( $j = 1, 2$ ) and those for which  $C \cdot L^+ = 0$ .

LEMMA 4.1. *If  $X$  contains no  $(-3)$ -curve, then  $\mathfrak{R}_0^1 = \mathfrak{R}_{-1}^2 = \emptyset$  and  $E$  is nef.*

*Proof.* If  $C$  is a smooth rational curve in  $\mathfrak{R}_0^1 \cup \mathfrak{R}_{-1}^2$ , then  $C \cdot K_X = 1$  and  $C^2 = -3$ . Hence the first assertion. The second one follows immediately from the Zariski decomposition of  $E$  in Proposition 3.6.  $\square$

From now on we also assume that  $X$  contains no  $(-3)$ -curves. With this hypothesis, our idea of comparison of  $E^2$  and  $E \cdot L^+$  takes the following form:

LEMMA 4.2.  $E^2 \geq E \cdot L^+$ .

Let us assume the lemma and complete the argument. First, notice that this inequality implies  $E^2 > 0$ . Then, by the Hodge index theorem,

$$E^2 \geq (L^+)^2 \geq L^2 \geq \frac{3}{2}(1 - \alpha_X) K_X^2,$$

where the last inequality comes from (2.3). Arguing as in the proof of Lemma 2.2, we obtain

$$E \cdot K_X \geq \sqrt{\frac{3}{2}(1 - \alpha_X) K_X^2} \quad \text{and} \quad L^+ \cdot K_X \geq \sqrt{\frac{3}{2}(1 - \alpha_X) K_X^2}.$$

Thus

$$K_X^2 \geq (L^+ + E) \cdot K_X \geq \sqrt{6(1 - \alpha_X)} K_X^2,$$

yielding  $\alpha_X \geq 5/6$ , which contradicts the hypotheses. At this stage Theorem 1.1 is proved and we turn to

*Proof of Lemma 4.2.* Combining (4.2), Lemma 4.1, and Corollary 3.8, we obtain

$$(E - Q) \cdot L^+ = \left( \sum_{C \in \mathfrak{R}_0^2} m_C C \right) \cdot L^+ \leq \sum_{C \in \mathfrak{R}_0^2} m_C, \quad (4.3)$$

where

$$\mathfrak{R}_0^2 = \{C \in \mathfrak{R}_0^2 \mid C \cdot L^+ > 0\}.$$

It is clear that the curves in  $\mathfrak{R}_0$  do not lie in  $L^-$ . Furthermore, by Corollary 3.8, every  $C$  in  $\mathfrak{R}_0$  intersects a unique irreducible component of type II. Denote this component by  $\Gamma_C$ . The correspondence  $C \mapsto \Gamma_C$  gives rise to the map

$$\rho : \mathfrak{R}_0 \longrightarrow \text{Irr}(E_{II}),$$

where  $\text{Irr}(E_{II})$  stands for the set of irreducible components of  $E_{II}$ . Using  $\rho$ , we will be able to replace the multiplicities  $m_C$  in (4.3) by the ones of the curves  $\Gamma \in \text{Irr}(E_{II})$ . This is done as follows.

Let  $\Gamma$  be a curve in  $\text{Irr}(E_{II})$  and set  $\mathfrak{R}_0(\Gamma) = \rho^{-1}(\Gamma)$ . Then for every  $C \in \mathfrak{R}_0(\Gamma)$  we have

$$0 = C \cdot E = C \cdot (E_I + E_{II}) = -4m_C + m_\Gamma,$$

since  $C^2 = -4$ ,  $C \cdot \Gamma = 1$ , and  $C$  intersects no other component of  $E$ . Thus  $m_\Gamma = 4m_C$ . Substituting it in (4.3), yields

$$(E - Q) \cdot L^+ \leq \sum_{C \in \mathfrak{R}_0} m_C = \frac{1}{4} \sum_{\Gamma \in \text{Irr}(E_{II})} n_\Gamma m_\Gamma, \quad (4.4)$$

where  $n_\Gamma = \text{card}(\rho^{-1}(\Gamma))$ .

CLAIM. If  $\Gamma$  is in  $\text{Irr}(E_{II})$ , then

$$\Gamma \cdot E \geq \frac{1}{4} n_\Gamma.$$

Assuming the claim, let us complete the proof of Lemma 4.2. We combine the inequality of the claim and (4.4) to obtain

$$(E - Q) \cdot L^+ \leq \frac{1}{4} \sum_{\Gamma \in \text{Irr}(E_{II})} n_\Gamma m_\Gamma \leq \sum_{\Gamma \in \text{Irr}(E_{II})} m_\Gamma \Gamma \cdot E = E_{II} \cdot E.$$

This and the definition of  $Q$  yield

$$E \cdot L^+ \leq Q \cdot L^+ + E_{II} \cdot E = \left( \sum_{C \in \mathfrak{R}_{\geq 1}^1} m_C C + \sum_{C \in \mathfrak{R}_{\geq 1}^2} m_C C \right) \cdot L^+ + E_{II} \cdot E.$$

Using Corollary 3.8 and the fact that  $E$  is nef (Lemma 4.1), we obtain

$$\begin{aligned} E \cdot L^+ &\leq \sum_{C \in \mathfrak{R}_{\geq 1}^1 \cup \mathfrak{R}_{\geq 1}^2} m_C + E_{II} \cdot E \\ &\leq \sum_{C \in \mathfrak{R}_{\geq 1}^1 \cup \mathfrak{R}_{\geq 1}^2} m_C C \cdot E + E_{II} \cdot E \\ &\leq E_I \cdot E + E_{II} \cdot E \\ &= E^2. \end{aligned}$$

□



*Proof of the claim.* We consider the divisor  $\left(\Gamma + \frac{1}{4} \sum_{C \in \mathfrak{R}_0(\Gamma)} C\right)$  and argue according to its self-intersection,

$$\left(\Gamma + \frac{1}{4} \sum_{C \in \mathfrak{R}_0(\Gamma)} C\right)^2 = \Gamma^2 + \frac{n_\Gamma}{4}.$$

If  $\Gamma^2 \leq -\frac{n_\Gamma}{4}$ , then we obtain

$$-\frac{n_\Gamma}{4} \geq \Gamma^2 \geq \Gamma \cdot L = \Gamma \cdot (K_X - E)$$

where the second inequality uses the fact that  $\Gamma$  is of type II and hence subject to the inequality (3.3). Thus we obtain

$$\Gamma \cdot E \geq \Gamma \cdot K_X - \Gamma^2 \geq \frac{n_\Gamma}{4} + \Gamma \cdot K_X > \frac{n_\Gamma}{4}.$$

If  $\Gamma^2 > -\frac{n_\Gamma}{4}$ , we use the fact that the divisor  $\left(\Gamma + \frac{1}{4} \sum_{C \in \mathfrak{R}_0(\Gamma)} C\right)$  enters  $E$  with multiplicity  $m_\Gamma$ , *i.e.*

$$E = m_\Gamma \left(\Gamma + \frac{1}{4} \sum_{C \in \mathfrak{R}_0(\Gamma)} C\right) + R_\Gamma,$$

where  $R_\Gamma$  is the residual part of  $E$ . This implies

$$\begin{aligned} \Gamma \cdot E &= m_\Gamma \Gamma \cdot \left(\Gamma + \frac{1}{4} \sum_{C \in \mathfrak{R}_0(\Gamma)} C\right) + \Gamma \cdot R_\Gamma \\ &\geq m_\Gamma \Gamma^2 + \frac{n_\Gamma m_\Gamma}{4} \end{aligned} \tag{4.5}$$

By (3.3),

$$\Gamma^2 \geq \Gamma \cdot L = \Gamma \cdot K_X - \Gamma \cdot E.$$

Substituting into (4.5),

$$\Gamma \cdot E \geq m_\Gamma \Gamma^2 + \frac{n_\Gamma m_\Gamma}{4} \geq m_\Gamma (\Gamma \cdot K_X - \Gamma \cdot E) + \frac{n_\Gamma m_\Gamma}{4}.$$

Solving for  $\Gamma \cdot E$ , we obtain

$$(m_\Gamma + 1) \Gamma \cdot E \geq m_\Gamma \Gamma \cdot K_X + \frac{n_\Gamma m_\Gamma}{4}.$$

Summing up with (4.5) leads to

$$(m_\Gamma + 2) \Gamma \cdot E \geq m_\Gamma (\Gamma \cdot K_X + \Gamma^2) + \frac{n_\Gamma m_\Gamma}{2} \tag{4.6}$$

and we continue to argue according to the sign of the expression  $(\Gamma \cdot K_X + \Gamma^2)$ .

*First case.* If  $\Gamma \cdot K_X + \Gamma^2 \geq 0$ , then the inequality (4.6) yields

$$\Gamma \cdot E \geq \frac{n_\Gamma m_\Gamma}{2(m_\Gamma + 2)} \geq \frac{n_\Gamma}{3} > \frac{n_\Gamma}{4},$$

where the second inequality follows from  $m_\Gamma = 4m_C \geq 4$ .

*Second case.* If  $\Gamma \cdot K_X + \Gamma^2 < 0$ , then  $\Gamma \cdot K_X + \Gamma^2 = -2$  and  $\Gamma$  is a smooth rational curve. Substituting this relation in (4.6) yields

$$(m_\Gamma + 2) \Gamma \cdot E \geq -2m_\Gamma + \frac{n_\Gamma m_\Gamma}{2}.$$

So

$$\Gamma \cdot E \geq \frac{m_\Gamma}{2(m_\Gamma + 2)} (n_\Gamma - 4). \quad (4.7)$$

Since  $\Gamma$  is a smooth rational curve,  $\Gamma^2 \leq -4$ —the surface  $X$  contains neither  $(-2)$ -curves nor  $(-3)$ -curves. Together with the assumption  $\Gamma^2 > -\frac{n_\Gamma}{4}$ , this inequality implies that  $n_\Gamma > 16$ . Using (4.7) and  $m_\Gamma \geq 4$ , we obtain

$$\Gamma \cdot E \geq \frac{1}{3} (n_\Gamma - 4) \geq \frac{n_\Gamma}{4} + \frac{n_\Gamma}{12} - \frac{4}{3} > \frac{n_\Gamma}{4}.$$

□

It is reasonable to guess that once the group  $H^1(\Theta_X(-m_0 K_X))$  vanishes for some  $m_0$ , the same should hold for all  $m > m_0$ . This is justified by the following:

**PROPOSITION 4.3.** *Let  $X$  be a smooth minimal complex surface of general type with  $p_g(X) \geq 1$  and  $\alpha_X < 1$ . Let  $c$  be a global section of  $\mathcal{O}_X(K_X)$  and let  $C = \{c = 0\}$  be the corresponding divisor. If  $C$  is reduced and irreducible, then the homomorphism*

$$H^1(\Theta_X(-(m+1)K_X)) \xrightarrow{c} H^1(\Theta_X(-mK_X))$$

*of the multiplication by  $c$  is injective for all  $m \geq 0$ .*

*Proof.* Let  $\xi$  be a non-zero class in  $H^1(\Theta_X(-(m+1)K_X))$  such that  $c \cdot \xi = 0$ . Using the identification

$$H^1(\Theta_X(-(m+1)K_X)) \cong \text{Ext}^1(\Omega_X(mK_X), \mathcal{O}_X(-K_X)),$$

we view  $\xi$  as the corresponding non-trivial extension class, *i.e.* the short exact sequence of vector bundles on  $X$

$$0 \longrightarrow \mathcal{O}_X(-K_X) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X(mK_X) \longrightarrow 0.$$

We dualize and consider the long exact sequence of cohomology groups

$$0 \longrightarrow H^0(\mathcal{T}_\xi^*) \longrightarrow H^0(\mathcal{O}_X(K_X)) \xrightarrow{\xi} H^1(\Theta_X(-mK_X)) \longrightarrow \dots$$

where the coboundary map is given by the cup-product with the extension class  $\xi$ . The assumption  $c \cdot \xi = 0$  implies that the section  $c$  is the image of a non-zero section  $s \in H^0(\mathcal{T}_\xi^*)$ . This gives rise to the following commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & & & (4.8) \\
 & & \downarrow & & & & \\
 & & \mathcal{O}_X & & & & \\
 & & \downarrow s & \searrow c & & & \\
 0 & \longrightarrow & \Theta_X(-mK_X) & \longrightarrow & \mathcal{T}_\xi^* & \longrightarrow & \mathcal{O}_X(K_X) \longrightarrow 0 \\
 & & & & \downarrow & & \\
 & & & & \text{coker}(s) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

Furthermore, since the divisor  $C = \{c = 0\}$  is reduced and irreducible, and the extension  $\xi$  is non-trivial, it follows that the sheaf  $\text{coker}(s)$  in (4.8) is torsion free. Dualizing again, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & & & (4.9) \\
 & & \downarrow & & & & \\
 & & \mathcal{F} & & & & \\
 & & \downarrow & \searrow \varphi_\xi & & & \\
 0 & \longrightarrow & \mathcal{O}_X(-K_X) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X(mK_X) \longrightarrow 0 \\
 & & \searrow \psi_\xi & & \downarrow & & \\
 & & & & \mathcal{J}_Z & & \\
 & & & & \downarrow & & \\
 & & & & 0 & & 
 \end{array}$$

where  $Z$  is the subscheme of zeros of  $s$ ,  $\mathcal{J}_Z$  is its sheaf of ideals, and  $\mathcal{F} = (\text{coker}(s))^*$  is a locally free sheaf of rank 2.

From (4.9) we deduce the Chern invariants of  $\mathcal{F}$ :

$$c_1(\mathcal{F}) = 2mK_X \quad \text{and} \quad c_2(\mathcal{F}) = c_2(X) + (m^2 - m - 1)K_X^2 - \deg(Z).$$

This and the assumption  $\alpha_X < 1$  imply

$$c_1^2(\mathcal{F}) - 4c_2(\mathcal{F}) = 4((m+1)K_X^2 - c_2(X)) + 4\deg(Z) > 0.$$

Thus  $\mathcal{F}$  is Bogomolov unstable and its Bogomolov destabilizing subsheaf  $\mathcal{O}_X(F)$  gives rise to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(F) \longrightarrow \mathcal{F} \longrightarrow \mathcal{J}_A(2mK_X - F) \longrightarrow 0, \quad (4.10)$$

where  $A$  is a 0-dimensional subscheme and  $\mathcal{J}_A$  its sheaf of ideals. The fact that  $\mathcal{O}_X(F)$  is the Bogomolov destabilizing sheaf also implies that the divisor

$$c_1(\mathcal{O}_X(F)) - \frac{1}{2} c_1(\mathcal{F}) = F - mK_X$$

is in the positive cone of  $NS(X)_{\mathbb{Q}}$ . But the morphism  $\varphi_{\xi}$  in (4.9) combined with (4.10) induces the monomorphism

$$\mathcal{O}_X(F - mK_X) \longrightarrow \Omega_X.$$

Hence  $\Omega_X$  has a subsheaf of rank 1 and Iitaka dimension 2 which is impossible.  $\square$

**COROLLARY 4.4.** *Let  $X$  be a smooth minimal surface of general type such that  $\alpha_X = c_2(X)/K_X^2 < 5/6$  and the canonical linear system  $|K_X|$  contains a reduced irreducible member. If  $X$  contains no smooth rational curves with self-intersection  $-2$  or  $-3$ , then*

$$H^1(\Theta_X(-mK_X)) = 0, \quad \text{for all } m \geq 1.$$

*Proof.* Follows immediately from Theorem 1.1 and Proposition 4.3.  $\square$

## 5 THE IRREGULARITY OF SURFACES IN $\mathbb{P}^4$

The aim of this section is to study the geography of surfaces of general type  $X$  in  $\mathbb{P}^4$  with fixed numerical invariants,  $d = \deg X$ ,  $K_X^2$ , and  $\chi(\mathcal{O}_X)$ . To this end we consider the set

$$\begin{aligned} \mathcal{S}_d(\chi, \beta) &:= \{X \mid X \subset \mathbb{P}^4 \text{ smooth surface of general type,} \\ &\deg X = d, \chi(\mathcal{O}_X) = \chi, K_X^2 = \beta\chi(\mathcal{O}_X)\}, \end{aligned} \quad (5.1)$$

where  $\beta \leq 9$ . There is a general result due to Decker and Schreyer, [5, Proposition 3.11] (see also [6, Proposition 3]), which establishes an upper bound on  $d$  for each given value of  $\chi$ . Here we will explore the irregularity of surfaces in  $\mathcal{S}_d(\chi, \beta)$ . More precisely, for the surfaces in the subset  $\mathcal{S}_d^{\circ}(\chi, \beta) \subset \mathcal{S}_d(\chi, \beta)$  defined by

$$\mathcal{S}_d^{\circ}(\chi, \beta) := \{X \in \mathcal{S}_d(\chi, \beta) \mid X \text{ minimal and with no irrational pencil}\}$$

we are able to show the following:

**THEOREM 5.1.** *Given a positive integer  $\chi$  and a rational  $\beta$ ,  $2 \leq \beta \leq 9$ , there exists a degree  $d(\chi, \beta)$  such that, if  $d > d(\chi, \beta)$ , then every surface  $X \in \mathcal{S}_d^{\circ}(\chi, \beta)$  has the irregularity at most 3.*

If the irregularity is equal to 3, our argument (see Lemma 5.4) shows that  $X$  must be contained in a quintic 3-fold in  $\mathbb{P}^4$  and must satisfy one of the following conditions:

(i)  $\Omega_X$  is generated by global sections, and  $K_X^2 = 6\chi(\mathcal{O}_X)$ ;

(ii) the canonical divisor has the form

$$K_X = \mu H + F + \sum_{C \cdot H = 2} m_C C + \sum_{l \cdot H = 1} m_l l,$$

where  $\mu \geq 1$ ,  $F$  is an effective divisor, and the sums run over collections of irreducible conics ( $C \cdot H = 2$ ) and lines ( $l \cdot H = 1$ ) contained in  $X$ .

A proof of this theorem is a consequence of several lemmas. Though the proofs of some of those results are somewhat lengthy and sometimes involve considerations of several cases, the main principle behind all the arguments is quite transparent: the exploiting of the *non-vanishing* of  $H^1(X, \Theta_X(-K_X))$ . This non-vanishing gives rise, as in Lemma 2.1, to a decomposition of the canonical divisor  $K_X$ . Then we seek to relate this decomposition to the linear system embedding  $X$  in  $\mathbb{P}^4$ .

The non-vanishing of  $H^1(X, \Theta_X(-K_X))$  turns out to be a cohomological version of the following result of Ellingsrud and Peskine in [6].

LEMMA 5.2. *Given the integer  $\chi$  and the rational  $\beta$ ,  $\beta \leq 9$ , there exists a degree  $d(\chi, \beta)$  such that for all  $d > d(\chi, \beta)$  and for every surface  $X$  in  $\mathbf{S}_d(\chi, \beta)$ , there is a 3-fold in  $\mathbb{P}^4$  of degree  $\leq 5$  containing  $X$ .*

*Proof.* If  $X$  is embedded in  $\mathbb{P}^4$  by the line bundle  $\mathcal{O}_X(H)$ , the numerical invariants of  $X$  satisfies the well-known formula (see e.g. [8], p.434),

$$d^2 - 10d - 5H \cdot K_X = 2K_X^2 - 12\chi(\mathcal{O}_X), \quad (5.2)$$

or, using the geometric genus of a hyperplane section,  $g_H$ , the adjunction formula, and the notation above,

$$d^2 - 5d - 10(g_H - 1) - 2(\beta - 6)\chi = 0. \quad (5.3)$$

By a result of Gruson and Peskine in [7] (the Halphen's bound), if  $d \geq 30$  and a hyperplane section  $H$  is not contained in a surface of degree 5, then

$$g_H - 1 \leq \frac{d^2 + 12d}{12}.$$

Putting this together with (5.3) yields

$$d^2 - 5d - 2(\beta - 6)\chi \leq \frac{5}{6}(d^2 + 12d),$$

or, equivalently,

$$d^2 - 90d - 12(\beta - 6)\chi \leq 0.$$

Set  $d(\chi, \beta) = 29$  if the discriminant of the left hand side is negative. Otherwise, we solve the inequality for  $d$  and set

$$d(\chi, \beta) = \max(29, 45 + \sqrt{45^2 + 12(\beta - 6)\chi}).$$

Then, if  $d > d(\chi, \beta)$ , the hyperplane sections of  $X$  must be contained in a surface of degree  $\leq 5$ . This and the classical result of Roth ([15], p.152, see also [6], (C), p.2) imply that  $X$  is contained in a 3-fold of degree  $\leq 5$  in  $\mathbb{P}^4$ , provided  $d > d(\chi, \beta)$ .  $\square$

Let  $\mathcal{J}_X$  be the ideal sheaf of  $X$  in  $\mathbb{P}^4$ . Set

$$m_X := \min\{m \in \mathbf{Z} \mid h^0(\mathcal{J}_X(m)) \neq 0\}, \quad (5.4)$$

*i.e.*  $m_X$  is the minimal degree of hypersurfaces in  $\mathbb{P}^4$  containing  $X$ .

Let  $\mathcal{N}_X$  be the normal bundle of  $X$  in  $\mathbb{P}^4$  and let  $\mathcal{N}_X^*$  be its dual, the conormal bundle. The normal bundle is a rank 2 bundle on  $X$  with determinant bundle  $\det(\mathcal{N}_X) = \wedge^2 \mathcal{N}_X = \mathcal{O}_X(K_X + 5H)$ , where  $H$  is a hyperplane section of  $X$ , *i.e.*  $\mathcal{O}_X(H) = \mathcal{O}_{\mathbb{P}^4}(1) \otimes \mathcal{O}_X$ . This implies

$$\mathcal{N}_X^* \cong \det(\mathcal{N}_X^*) \otimes \mathcal{N}_X = \mathcal{N}_X(-K_X - 5H). \quad (5.5)$$

By the definition of  $m_X$  in (5.4) and the identity  $\mathcal{N}_X^* = \mathcal{J}_X / \mathcal{J}_X^2$ , we deduce  $h^0(\mathcal{N}_X^*(m_X)) \neq 0$ . This together with the identification in (5.5) yield

$$H^0(\mathcal{N}_X^*(m_X)) = H^0(\mathcal{N}_X(-K_X - (5 - m_X)H)) \neq 0.$$

This non-vanishing is related to the  $H^1$  of the tangent bundle  $\Theta_X$  via the normal sequence of  $X$  in  $\mathbb{P}^4$ ,

$$0 \longrightarrow \Theta_X \longrightarrow \Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X \longrightarrow \mathcal{N}_X \longrightarrow 0.$$

Tensoring with  $\mathcal{O}_X(-K_X - (5 - m_X)H)$  and passing to the long exact sequence of cohomology groups, we obtain

$$H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-D)) \longrightarrow H^0(\mathcal{N}_X(-D)) \xrightarrow{\delta_X} H^1(\Theta_X(-D)), \quad (5.6)$$

where we set  $D = K_X + (5 - m_X)H$ . In view of Lemma 5.2, we are interested in the case  $m_X \leq 5$ . The following result is a cohomological interpretation of that lemma.

**LEMMA 5.3.** *If  $m_X \leq 5$ , then the coboundary homomorphism  $\delta_X$  in (5.6) is injective unless  $m_X = 5$ ,  $p_g(X) \leq 2$ , and the degree of  $X$  in  $\mathbb{P}^4$  is at most 16.*

*Proof.* The failure of injectivity of  $\delta_X$  in (5.6) implies the non-vanishing of

$$H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-D)) = H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-K_X - (5 - m_X)H)).$$

Using the Euler sequence on  $\mathbb{P}^4$ ,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(1))^* \otimes \mathcal{O}_{\mathbb{P}^4}(1) \longrightarrow \Theta_{\mathbb{P}^4} \longrightarrow 0,$$

it follows, by restriction to  $X$ , that either  $H^0(\mathcal{O}_X(-K_X - (4 - m_X)H))$  or  $H^1(\mathcal{O}_X(-K_X - (5 - m_X)H))$  must be nonzero. But for  $m_X \leq 4$  both groups vanish: the first one because  $X$  is of general type, and the second one because the divisor  $(K_X + (5 - m_X)H)$  is nef and big. Hence, we must have  $m_X = 5$ , *i.e.*  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-K_X)) \neq 0$ .

From the Euler sequence, this non-vanishing can occur either if  $H^0(\mathcal{O}_X(H - K_X)) \neq 0$ , or if

$$\ker \left( H^1(\mathcal{O}_X(-K_X)) \longrightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(H - K_X)) \right) \neq 0. \quad (5.7)$$

We consider each of these two cases.

*First case.* If  $H^0(\mathcal{O}_X(H - K_X)) \neq 0$ , then

$$\begin{aligned} & \text{either } H = K_X, \\ & \text{or } H - K_X = \Gamma, \quad \text{with } \Gamma \text{ a non-zero effective divisor.} \end{aligned} \quad (5.8)$$

We claim that the former is impossible. Indeed, if  $H = K_X$ , then by [1],  $X$  is a complete intersection of two hypersurfaces in  $\mathbb{P}^4$ . Since  $m_X = 5$ , it follows that the degrees of these hypersurfaces are 5 and  $n \geq 5$ . But then, by the adjunction formula for  $K_X$ , we have  $K_X = nH$ , which contradicts  $H = K_X$ . Thus (5.8) holds and we deduce

$$p_g(X) = h^0(\mathcal{O}_X(K_X)) = h^0(\mathcal{O}_X(H - \Gamma)) \leq 3. \quad (5.9)$$

Furthermore, the inequality must be strict, *i.e.*  $p_g(X) \leq 2$ . Indeed, if the equality holds in (5.9), then  $\Gamma$  is a line in  $\mathbb{P}^4$ . Intersecting both sides in (5.8) with the line  $\Gamma$ , we obtain

$$1 = H \cdot \Gamma = (K_X + \Gamma) \cdot \Gamma = -2,$$

which is absurd. Thus  $p_g(X) \leq 2$  and it remains to bound the degree  $d$  of  $X$ . From (5.8),

$$H \cdot K_X = H^2 - \Gamma \cdot H < d.$$

Substituting this in the relation (5.3), we obtain

$$d^2 - 15d < 2K^2 - 12\chi(\mathcal{O}_X). \quad (5.10)$$

Since  $p_g(X) \leq 2$ , it follows  $\chi(\mathcal{O}_X) \leq 3$ . This and the Bogomolov-Miyaoka-Yau inequality imply

$$2K^2 - 12\chi(\mathcal{O}_X) \leq 18.$$

Putting this together with (5.10), we obtain  $d \leq 16$ .

*Second case.* If  $\ker \left( H^1(\mathcal{O}_X(-K_X)) \longrightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(H - K_X)) \right) \neq 0$ , let  $X_0$  be the minimal model of  $X$ . Since  $X$  is of general type,  $X_0$  is uniquely defined. Let

$$f : X \longrightarrow X_0$$

be the corresponding sequence of blowing-down maps. The canonical divisor of  $X$  can be written as follows:

$$K_X = f^*K_{X_0} + \Delta \quad (5.11)$$

where  $\Delta$  is the exceptional divisor of  $f$ . In particular,  $\Delta$  is composed of rational curves on  $X$  contracted to points by  $f$ . Thus  $\Delta \cdot f^*K_{X_0} = 0$  and  $f^*K_{X_0}$  is nef and big— $X_0$  is minimal and of general type.

The non-vanishing of  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-K_X))$  implies the non-vanishing of  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-f^*K_{X_0}))$ . From the Euler sequence and the fact that  $f^*K_{X_0}$  is nef and big, the latter non-vanishing yields

$$H^0(\mathcal{O}_X(H - f^*K_{X_0})) \neq 0.$$

As before, there are two possibilities, either  $H = f^*K_{X_0}$ , or  $H - f^*K_{X_0} = \Gamma$ , with  $\Gamma$  effective and nonzero. Since  $H$  is very ample, the first possibility yields  $X_0 = X$  and  $H = K_X$  which implies the vanishing of  $H^1(\mathcal{O}_X(-K_X))$ , contrary to our assumption. Hence the latter possibility must hold.

Arguing as in the first case, we deduce  $p_g(X) \leq 3$ . Furthermore, the inequality must be strict. Indeed, if  $p_g(X) = 3$ , then  $\Gamma$  is a line. Since  $H - f^*K_{X_0} = \Gamma$ , it follows that

$$H = K_X + \Gamma - \Delta. \quad (5.12)$$

Intersecting the both sides with the line  $\Gamma$  yields

$$1 = H \cdot \Gamma = (K_X + \Gamma) \cdot \Gamma - \Delta \cdot \Gamma = -2 - \Delta \cdot \Gamma.$$

Hence  $\Delta \cdot \Gamma = -3$  and the line  $\Gamma$  is a component of  $\Delta$ , *i.e.*

$$\text{the divisor } (\Delta - \Gamma) \text{ is effective.} \quad (5.13)$$

On the other hand, the condition (5.7) implies that for every  $h \in H^0(\mathcal{O}_X(H))$ , the cup-product

$$H^1(\mathcal{O}_X(-K_X)) \xrightarrow{h} H^1(\mathcal{O}_X(H - K_X))$$

has a nontrivial kernel. This implies that  $H^0(\mathcal{O}_C(H - K_X)) \neq 0$ , for every  $C$  in the linear system  $|H|$ . By (5.12),  $H - K_X = \Gamma - \Delta$ , hence

$$H^0(\mathcal{O}_C(\Gamma - \Delta)) \neq 0 \text{ for every } C \in |H|. \quad (5.14)$$

This, together with (5.13), implies  $\Delta = \Gamma$ . Substituting it in (5.12) leads to  $H = K_X$  which is impossible. Thus  $p_g(X) \leq 2$  and it remains to give the upper bound on  $d$ .

From (5.14), it follows that  $H \cdot (\Gamma - \Delta) \geq 0$ . This and (5.12) yield  $H \cdot K_X \leq H^2 = d$  and the rest of the argument is as in the first case.  $\square$

We now bring in an additional hypothesis:

$$X \text{ is an irregular surface of general type with no irrational pencil.} \quad (5.15)$$

This hypothesis combined with the assumption  $m_X \leq 5$  yield the following characterization.

LEMMA 5.4. *Let  $m_X \leq 5$  and assume  $X$  to be subject to (5.15). Then  $m_X = 4$  or  $5$ . Furthermore,*

- *if  $m_X = 4$ , then  $q(X) = 2$  and*

$$K_X = \mu H + F + \sum_{C \cdot H=2} m_C C + \sum_{l \cdot H=1} m_l l \quad (5.16)$$

*with  $\mu \geq 1$ ,  $F$  an effective divisor, and the sums run over collections of irreducible conics ( $C \cdot H = 2$ ) and lines ( $l \cdot H = 1$ ) contained in  $X$ ;*

- *if  $m_X = 5$  and  $X$  is minimal, then  $q(X) \leq 3$ . Moreover, if equality holds, then one of the following can occur:*



(i)  $\Omega_X$  is generated by global sections, and  $K_X^2 = 6\chi(\mathcal{O}_X)$ .

(ii) The canonical divisor has the following form

$$K_X = \mu H + F + \sum_{C \cdot H=2} m_C C + \sum_{l \cdot H=1} m_l l,$$

where  $\mu \geq 1$ ,  $F$  is an effective divisor, and the sums are as in (5.16).

REMARK. It is to be noticed that even if the coefficient  $\mu$  above satisfies the same condition in both cases, these conditions occur for different geometric reasons. The proof will show that if  $m_X = 4$ , then  $K_X = E$  and  $\mu$  corresponds to the sum,  $\mu_{II}$ , of the coefficients of type II components of  $E$ , whereas if  $m_X = 5$  then  $L = H$ , i.e.  $K_X = H + E$ , and  $\mu = \mu_{II} + 1$ .

*Proof.* The proof will be done in several steps. We begin by showing in *Step 1* that  $m_X = 4$  or 5. Then, we go on with the detailed study of each value in *Step 2* and *Step 3*.

*Step 1.* Note that the hypothesis (5.15) implies that the irregularity  $q(X) \geq 2$ . This and the fact that  $X$  is of general type yield  $p_g(X) \geq 2$  as well. Then, according to Lemma 5.3, the coboundary homomorphism  $\delta_X$  in (5.6) is injective unless  $m_X = 5$ ,  $p_g(X) \leq 2$ , and the degree of  $X$  in  $\mathbb{P}^4$  is at most 16. Hence, in this situation,  $p_g(X) = q(X) = 2$ . Thus we may assume that  $\delta_X$  is injective, hence that  $H^1(\Theta_X(-K_X - (5 - m_X)H)) \neq 0$ .

Let  $\eta$  be a nonzero section of  $\mathcal{N}(-K_X - (5 - m_X)H)$  and let  $\xi = \delta_X(\eta)$  be its image in  $H^1(\Theta_X(-K_X - (5 - m_X)H))$ . Viewing it as an extension class in  $\text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X - (5 - m_X)H))$  via the natural identification

$$H^1(\Theta_X(-K_X - (5 - m_X)H)) \cong \text{Ext}^1(\Omega_X, \mathcal{O}_X(-K_X - (5 - m_X)H)),$$

we obtain the following exact sequence of locally free sheaves on  $X$ ,

$$0 \longrightarrow \mathcal{O}_X(-K_X - (5 - m_X)H) \longrightarrow \mathcal{T}_\xi \longrightarrow \Omega_X \longrightarrow 0. \quad (5.17)$$

Observe that the divisor  $(K_X + (5 - m_X)H)$  is nef and big for  $m_X \leq 5$ . This implies that  $H^0(\mathcal{T}_\xi) = H^0(\Omega_X)$  and we consider the smallest saturated subsheaf  $\mathcal{F}$  of  $\mathcal{T}_\xi$  containing the image of the evaluation map

$$H^0(\mathcal{T}_\xi) \otimes \mathcal{O}_X \longrightarrow \mathcal{T}_\xi.$$

The hypothesis (5.15) implies that the rank of  $\mathcal{F}$  is at least 2. On the other hand, from the defining sequence (5.17), it follows that  $\det(\mathcal{T}_\xi) = \mathcal{O}_X(-(5 - m_X)H)$ . This implies that  $\mathcal{F}$  has rank 2 unless  $m_X = 5$ . In this case, with the additional assumption of  $X$  being minimal, we still have  $\mathcal{F}$  of rank 2, unless  $\mathcal{F} = \mathcal{T}_\xi = \oplus^3 \mathcal{O}_X$  (these equalities follow immediately from  $\det(\mathcal{T}_\xi) = \mathcal{O}_X$ ). This yields the case (i) of the lemma.

Thus we may assume that  $\mathcal{F}$  has rank 2 and we arrive at a diagram analogous to the one in (2.1),

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathcal{F} & & & \\
& & & \downarrow & \searrow \varphi_\xi & & \\
0 & \longrightarrow & \mathcal{O}_X(-K_X - (5 - m_X)H) & \longrightarrow & \mathcal{T}_\xi & \longrightarrow & \Omega_X \longrightarrow 0 \\
& & \searrow \psi_\xi & & \downarrow & & \\
& & & & \mathcal{J}_Z(-L - (5 - m_X)H) & & \\
& & & & \downarrow & & \\
& & & & 0 & & 
\end{array}
\tag{5.18}$$

where  $\mathcal{O}_X(L) = \det(\mathcal{F})$ . By construction, the morphism  $\varphi_\xi$  is generically an isomorphism. Arguing as in the proof of Lemma 2.1, we obtain the decomposition of the canonical divisor

$$K_X = L + E \tag{5.19}$$

where  $L$  is given by the determinant of  $\mathcal{F}$  and  $E = c_1(\text{coker}(\varphi_\xi))$  is a non-zero effective divisor. We also observe that the hypothesis (5.15) implies that  $L$  is effective as well.

Let  $e$  be a section of  $\mathcal{O}_X(E)$  corresponding to the divisor  $E$ . Similarly to the statement of Lemma 2.4, we have that  $e$  annihilates the cohomology class  $\xi$ , *i.e.*

$$e \cdot \xi = 0 \quad \text{in} \quad H^1(\Theta_X(E - K_X - (5 - m_X)H)). \tag{5.20}$$

From the commutative diagram

$$\begin{array}{ccccc}
H^0(\mathcal{N}_X(-D)) & \xrightarrow{\delta_X} & H^1(\Theta_X(-D)) & & \\
\downarrow e & & \downarrow e & & \\
H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(E - D)) & \longrightarrow & H^0(\mathcal{N}_X(E - D)) & \xrightarrow{\delta_X(E)} & H^1(\Theta_X(E - D))
\end{array}
\tag{5.21}$$

where  $D = K_X + (5 - m_X)H$ , it follows that

$$\delta_X(E)(e \cdot \eta) = e \cdot \delta_X(\eta) = e \cdot \xi = 0,$$

where the last equality is (5.20). From this and the diagram (5.21), it follows that the non-zero section  $e \cdot \eta$  of  $\mathcal{N}_X(E - D)$  comes from  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(E - D))$ . Thus, using the decomposition (5.19), we obtain

$$H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L - (5 - m_X)H)) = H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(E - K_X - (5 - m_X)H)) \neq 0.$$

From the Euler sequence of  $\Theta_{\mathbb{P}^4}$ , the non-vanishing of  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L - (5 - m_X)H))$  may occur for one of the following reasons: either

$$H^0(\mathcal{O}_X(-L - (4 - m_X)H)) \neq 0, \quad (5.22)$$

or

$$\ker \left( H^1(\mathcal{O}_X(-L - (5 - m_X)H)) \rightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(-L - (4 - m_X)H)) \right) \neq 0. \quad (5.23)$$

It is obvious that (5.22) implies that  $m_X \geq 4$ . We claim that the same condition is necessary for (5.23). Indeed, let  $h$  be a nonzero section of  $\mathcal{O}_X(H)$  and let  $C_h$  be the corresponding divisor in the linear system  $|H|$ . Then, from the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(-L - (5 - m_X)H) \longrightarrow \mathcal{O}_X(-L - (4 - m_X)H) \longrightarrow \\ \longrightarrow \mathcal{O}_{C_h}(-L - (4 - m_X)H) \longrightarrow 0, \end{aligned}$$

it follows that (5.23) forces the non-vanishing of the cohomology group  $H^0(\mathcal{O}_{C_h}(-L - (4 - m_X)H))$ . For this to hold, one must again have  $m_X \geq 4$ . This completes the proof of the first assertion of the lemma. In the rest of the proof we consider separately the two possible values of  $m_X$ .

*Step 2.* If  $m_X = 4$ , then the above argument shows that for (5.22) (respectively for (5.23)) to hold,  $L$  must be 0. This together with the definition of the sheaf  $\mathcal{F}$  in (5.18) imply

$$\mathcal{F} = \oplus^2 \mathcal{O}_X.$$

In particular,  $q(X) = 2$ .

Next we turn to the formula for  $K_X$  in (5.16). For this, we use the decomposition of  $K_X$  given in (5.19). Due to  $L = 0$ , this simply reads

$$K_X = E = E_I + E_{II},$$

where the last equality is as in (3.4). Furthermore from the diagram (5.18) with  $m_X = 4$ , the conditions characterizing the curves of type I and type II become

$$\textbf{type I:} \quad \deg Z_C \leq \deg \eta_C^* Z_C = 2 - 2g(\tilde{C}) - H \cdot C \quad (5.24)$$

$$\textbf{type II:} \quad H^0(\mathcal{O}_C(C - H)) \neq 0 \quad (5.25)$$

where  $\eta_C : \tilde{C} \rightarrow C$ , in (5.24), denotes the normalization of  $C$ .

From (5.24), it follows that the irreducible components of  $E_I$  are either lines,  $C \cdot H = 1$ , or conics,  $C \cdot H = 2$ . From (5.25), it follows that  $C - H = F_C$  is an effective divisor. Hence, the canonical divisor  $K_X$  satisfies

$$K_X = E = \mu_{II}H + \sum_{\substack{C \text{ of} \\ \text{type II}}} m_C F_C + \sum_{C \cdot H=2} m_C C + \sum_{l \cdot H=1} m_l l, \quad (5.26)$$

where  $m_D$  stands for the multiplicity of a component  $D$  in  $K_X$  and

$$\mu = \mu_{II} = \sum_{C \text{ of type II}} m_C.$$

This formula contains all the ingredients of (5.16) except the assertion about  $\mu_{II}$ . To see that, let us assume that  $\mu_{II} = 0$ . Then

$$K_X = \sum_{C \cdot H=2} m_C C + \sum_{l \cdot H=1} m_l l,$$

*i.e.* the canonical divisor is composed entirely of rational curves. Thus it is contracted to points by the Albanese map and hence, the Zariski decomposition of  $K_X$  has no positive part, contradicting the fact that  $X$  is of general type.

*Step 3.* If  $m_X = 5$  and  $X$  is minimal, then the construction of the subsheaf  $\mathcal{F}$  in (5.18) goes through and we may assume it to be of rank 2 (otherwise the argument in *Step 1* shows that (i) of the lemma holds). This gives the decomposition of the canonical divisor  $K_X$  as in (5.19). Continuing the argument as in *Step 1*, we arrive at

$$H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L)) \neq 0. \quad (5.27)$$

At this point we assume that the irregularity  $q(X) \geq 3$  (otherwise there is nothing to prove). Consider the homomorphism

$$\wedge^2 H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{O}_X(L)). \quad (5.28)$$

The hypothesis (5.15) and the argument in the lemma of Castelnuovo-de Franchis, see [2, Proposition X.9], imply

$$h^0(\mathcal{O}_X(L)) \geq 2q(X) - 3 \geq 3. \quad (5.29)$$

We claim that this leads to

$$\mathcal{O}_X(L) = \mathcal{O}_X(H). \quad (5.30)$$

Let us assume this and complete our argument. The inequality (5.29) combined with (5.30) yield

$$2q(X) - 3 \leq h^0(\mathcal{O}_X(L)) = h^0(\mathcal{O}_X(H)) = 5, \quad (5.31)$$

*i.e.*  $q(X) \leq 4$ . We will now rule out the case  $q(X) = 4$ . Indeed, if  $q(X) = 4$  holds, the homomorphism in (5.28) has the form

$$\wedge^2 H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{O}_X(H)) \quad (5.32)$$

and must be surjective with a 1-dimensional kernel. Hence  $\mathcal{F}$  is generated by global sections and it maps  $X$  into the Grassmannian  $\text{Gr}(1, 3) = \text{Gr}(1, \mathbb{P}(H^0(\mathcal{F})^*))$  of lines in  $\mathbb{P}(H^0(\mathcal{F})^*) = \mathbb{P}^3$ . Composing this map with the Plücker map of the Grassmannian, we get

$$X \longrightarrow \text{Gr}(1, 3) \hookrightarrow \mathbb{P}(\wedge^2 H^0(\mathcal{F})^*) = \mathbb{P}^5$$

Furthermore  $X \subset \mathbb{P}^4$  is contained in the intersection of the Plücker embedding of the Grassmannian  $\text{Gr}(1, 3)$  with the hyperplane in  $\mathbb{P}(\wedge^2 H^0(\mathcal{F})^*)$  corresponding to the kernel of the homomorphism in (5.32). The image of  $\text{Gr}(1, 3)$  under the Plücker map is a quadric. Hence  $X \subset \mathbb{P}^4$  is contained in a quadric which is impossible by the first assertion of the lemma.

Thus  $q(X) = 3$  and the decomposition in (5.19) reads as follows:

$$K_X = H + E = H + E_{II} + E_I \quad (5.33)$$

where we use again the decomposition of  $E$  into types. Furthermore, the irreducible components of  $E$  are subject to the same conditions as in (5.24) and (5.25). This leads to the formula for  $E$  as in (5.26). Substituting it into (5.33) yields

$$K_X = (\mu_{II} + 1)H + \sum_{\substack{C \text{ of} \\ \text{type II}}} m_C F_C + \sum_{C \cdot H = 2} m_C C + \sum_{l \cdot H = 1} m_l l$$

where the notation have the same meaning as in (5.26).  $\square$

*Proof of (5.30).* For  $m_X = 5$  the conditions (5.22) and (5.23) become respectively  $H^0(\mathcal{O}_X(H - L)) \neq 0$  and

$$\ker \left( H^1(\mathcal{O}_X(-L)) \longrightarrow H^0(\mathcal{O}_X(H))^* \otimes H^1(\mathcal{O}_X(H - L)) \right) \neq 0. \quad (5.34)$$

CLAIM.  $H^0(\mathcal{O}_X(H - L)) \neq 0$ .

This claim implies the equality (5.30). Indeed, assume  $H - L = \Gamma \neq 0$ . Then we have

$$h^0(\mathcal{O}_X(H - \Gamma)) = h^0(\mathcal{O}_X(L)) \geq 3,$$

where the last inequality comes from (5.29). This implies that the equality must hold and that  $\Gamma$  is a line in  $\mathbb{P}^4$ . Thus

$$H = L + \Gamma = K - E + \Gamma \quad (5.35)$$

Taking the intersection with  $\Gamma$ , we have

$$1 = \Gamma \cdot H = \Gamma \cdot (K_X + \Gamma) - \Gamma \cdot E = -2 - \Gamma \cdot E.$$

From this it follows that  $E \cdot \Gamma = -3$  which contradicts Remark 3.5.

We now turn to the proof of the claim,  $H^0(\mathcal{O}_X(H - L)) \neq 0$ . From (5.29), it follows that

$$L = M + F \quad (5.36)$$

where  $M$  (resp.  $F$ ) is the moving (resp. fixed) part of  $L$ . In particular,  $M$  is nef, and since  $X$  has no irrational pencil, it is also big. The non-vanishing of the group  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L))$  (see (5.27)) implies that

$$H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-M)) \neq 0$$

as well. From the Euler sequence of  $\Theta_{\mathbb{P}^4}$  it follows that this group is the middle term of the exact sequence

$$H^0(\mathcal{O}_X(H))^* \otimes H^0(\mathcal{O}_X(H - M)) \longrightarrow H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-M)) \longrightarrow H^1(\mathcal{O}_X(-M)).$$

The fact that  $M$  is nef and big yields  $H^1(\mathcal{O}_X(-M)) = 0$ . Hence  $H^0(\mathcal{O}_X(H - M)) \neq 0$ . Using this non-vanishing and the decomposition (5.36),  $L = M + F$ , we see that

$$L = H + D, \quad (5.37)$$

with  $D$  an effective divisor. Indeed,  $H^0(\mathcal{O}_X(H - M)) \neq 0$  if either  $M = H$ , and  $D = F$  in (5.37), or  $H - M = l$ , with  $l$  a line since  $h^0(\mathcal{O}_X(H - l)) = h^0(\mathcal{O}_X(M)) = h^0(\mathcal{O}_X(L)) \geq 3$ . Now  $l \cdot E \geq -2$  (see Remark 3.5), hence

$$0 \geq l \cdot (K_X + l - E) = l \cdot (L + l) = l \cdot (H + L - M) = l \cdot (H + F) = 1 + l \cdot F.$$

It follows that  $l$  is contained in  $F$  and we take  $D = F - l$  to obtain the decomposition in (5.37).

We now use (5.37) to show that the kernel in (5.23) vanishes. Indeed, a non-trivial element of this kernel gives rise to the non-trivial kernel for the cup product

$$H^1(\mathcal{O}_X(-L)) \xrightarrow{h} H^1(\mathcal{O}_X(H - L))$$

for every  $h \in H^0(\mathcal{O}_X(H))$ . Thus  $H^0(\mathcal{O}_{C_h}(H - L)) \neq 0$ , where  $C_h = \{h = 0\}$  is the divisor corresponding to  $h$ . Substituting for  $L$  the expression (5.37), we deduce that  $H^0(\mathcal{O}_C(-D)) \neq 0$  for every divisor  $C$  in the linear system  $|H|$ . Since  $D$  is effective, we obtain that  $D = 0$  and thus  $L = H$ . This implies the vanishing of  $H^1(\mathcal{O}_X(-L))$  in (5.34), hence  $H^0(\mathcal{O}_X(H - L)) \neq 0$ .  $\square$

*Proof of Theorem 5.1.* Let  $d(\chi, \beta)$  be as in Lemma 5.2 and let  $X \in \mathcal{S}_d^o(\chi, \beta)$ . By Lemma 5.2,  $X$  is contained in a 3-fold of degree  $m_X \leq 5$ . From Lemma 5.4 it follows that  $m_X$  is either 4 or 5, and that  $q(X) \leq 3$ . Furthermore, if  $q(X) = 3$ , then (i) or (ii) of Lemma 5.4 must hold.  $\square$

REMARK. The well-known example of Horrocks and Mumford in [9] is essentially the only known surface in  $\mathbb{P}^4$  of irregularity 2. Though Theorem 5.1 does not rule out the possibility of surfaces in  $\mathbb{P}^4$  with irregularity 3, it shows, following [4, Proposition 4.1], that such a hypothetical surface<sup>2</sup> could be a divisor in an Abelian variety of dimension 3.

Using similar reasoning, we can restrict the topology of surfaces of high degree and bounded holomorphic Euler characteristic in  $\mathbb{P}^4$ .

THEOREM 5.5. *Given a positive integer  $n$ , every surface  $X$  in  $\mathcal{S}_d(\chi, \beta)$  has negative topological index, i.e.  $\alpha_X > 1/2$ , provided  $d > d(\chi, \beta)$ . (The notation  $\mathcal{S}_d(\chi, \beta)$  and  $d(\chi, \beta)$  are to be found in (5.1) and Lemma 5.2 respectively.)*

*Proof.* By Lemma 5.2,  $X$  is contained in a 3-fold of degree  $m_X \leq 5$ . Following the arguments in Lemma 5.3 and Lemma 5.4, we deduce that  $H^1(\Theta_X(-K_X)) \neq 0$ . This implies the decomposition of  $K_X = L + E$  and that  $H^0(\Theta_{\mathbb{P}^4} \otimes \mathcal{O}_X(-L)) \neq 0$ . The latter condition yields

$$H \cdot L \leq H^2 = d. \quad (5.38)$$

CLAIM. If  $\alpha_X \leq 1/2$ , then  $E \cdot H \leq L \cdot H$ .

To justify the claim, we assume the opposite,  $E \cdot H > L \cdot H$ . From this and Lemma 2.3, 1), we deduce

$$(E - L) \cdot (K_X + \lambda H) = 0$$

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<sup>2</sup>The condition (ii) of Lemma 5.4 could be envisaged as a degenerate case of (i).

for some  $\lambda > 0$ . By the Hodge index and Lemma 2.3, 2), it follows that  $E$  is numerically equivalent to  $L$ , which contradicts Lemma 2.3, 1).

To end the proof of the theorem, we combine the claim and (5.38) to deduce

$$H \cdot K_X = H \cdot (L + E) \leq 2H \cdot L \leq 2d.$$

This inequality and (5.2) yield

$$2d \geq H \cdot K_X = \frac{1}{5}(d^2 - 10d - 2(\beta - 6)\chi).$$

Hence, when the corresponding discriminant is not negative,

$$d \leq 10 + \sqrt{100 + 2(\beta - 6)\chi} < d(\chi, \beta).$$

□

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